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Nuclear Algebraic Geometry and *SO*(10)

J. A. de Wet[1]

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In this contribution nuclear representations of the Dirac ring, developed over many years, are shown to be a particular case of a theorem in algebraic geometry which at the same time associates them with a Hodge decomposition of a Kaehler manifold. This yields a shape that in some cases is independent of any appeal to a symmetry group. However, because the nuclear representations are in the infinitesimal ring of $SO(4)$ and the internal space of each representation is in a Kaehler (even Calabi-Yau) manifold K; the group $SO(10) = SO(4) \times K$ can give additional information. This paper develops the very fruitful symbiosis between algebra and irreducible representations of $SO(10)$ and covers some aspects of string theory.

KEY WORDS: Calabi-Yau manifolds; strong fields; Hodge theory; nuclear algebraic surfaces; IR's of *SO*(10).

1. INTRODUCTION

Long ago Eddington and Dirac asociated nucleons and electrons with representations of the centralizer D of the quaternion or Dirac ring. Algebraists have also shown that these representations lie in a Kaehler manifold with a Hodge decomposition which is associated with an abelian variety (or polynomial) that yields a shape (see for example, Griffiths, 1969; Moonen and Zarhin, 1999). Following Eddington's lead the Author (de Wet, 1998) has been able to find an irreducible representation of D with the operators of spin, isospin, and parity carried by a nucleon, and therefore to incorporate the many-nucleon case by constructing the tensor product. Odd A nuclei have an internal Kaehler (even Calabi-Yau) manifold and as expected exhibit not only mirror symmetry but decompose beautifully into Hodge classes from which nuclear shapes may be determined.

Although algebraists prefer not to work with matrix representations much may be gained from this approach. One thinks of Pauli matrices for SU_2 and representations of quaternions, given for example by Eddington (1953), which were employed by de Wet (1973, 1998) to find the well-known angular momentum operators for a coupled system of P protons and N neutrons which together with conjugate parity operators π in (2.4) below constitute the six generators of $O(4)$. However, these

[1] P.O. Box 514, Plettenberg Bay 6600, South Africa; e-mail: jadew@global.co.za.

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relations also follow from the Z_2-grading of the algebra as shown by Lawson and Michelsohn (1989), so there is an intimate connection between nuclear algebraic geometry and $SO(10)$, because the spinorial representations of D in $O(4)$ lie in a Calabi-Yau space K. In this contribution the symbiosis will be used to analyse nuclear structure. For example, Fig. 1 shows strings of electric flux lines binding the rotating and spinning nucleons of ^{9}Be and ^{9}Li. These are equivalent to geodesics on the nuclear manifolds computed by the matrix representations introduced in Section 2. In contrast Fig. 2 is an idealized quintic hypersurface (or 2-brane) in 4-space which also carries a string (1-brane) on ^{9}Li and does not depend on a matrix representation. A four-dimensional view of this three-dimensional section appears in Greene (1999) while Fig. 2 is taken from a program written by Hanson (1994).

Unfortunately there is not a unique relationship between algebraic varieties, or hypersurfaces, and the associated Hodge decomposition, because a hypersurface is generated by the poles, or singularities at the origin which are absent in the case of the stable nuclei ^{9}Be, ^{11}C, ^{13}C (and their mirror partners) investigated so far by the methods of Section 2 (where a Kaehler metric will be found). This metric is twisted in the case of the unstable nucleus ^{9}Li and has a pole of Order 5 at the origin (see Fig. 4) which following Griffiths (1969) will generate a quintic in a four-dimensional complex projective space. The twisted 2-branes meeting in a black hole at the origin of Fig. 2 could be the source of elementary particles according to Greene *et al.* (1995).

The contact with nuclear theory is provided by the labeling of a partition $A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ of a nuclear canonical ensemble that yields the states:

$\lambda_1 =$ number of neutrons with positive spin and negative parity,
$\lambda_2 =$ number of neutrons with negative spin and positive parity,
$\lambda_3 =$ number of protons with negative spin and negative parity,
$\lambda_4 =$ number of protons with positive spin and positive parity.

In this way a row of an irreducible representation is labeled by $[\lambda] \equiv [\lambda_1\lambda_2\lambda_3\lambda_4]$.

The spin and parity are respectively

$$s = \frac{1}{2}(A - 2(\lambda_2 + \lambda_3)), \quad p = \frac{1}{2}(2(\lambda_2 + \lambda_4) - A) \tag{1.1}$$

and it is possible to find the eigenvalues and hence wave functions and metric of a Hodge decomposition of an irreducible three-form $C_{[\lambda]}$, in the centralizer D, by direct substitution into (2.7a) with the identifications $\sigma_o = 2is$, $\pi_o = 2ip$. The fact that these eigenvalues agree precisely, up to sign, with those of a matrix representation justifies the canonical labeling. However, to find the signs of the states labeling the rows of $C_{[\lambda]}$ an irreducible matrix representation of a subspace μ is mandatory.

A representation of D has, by construction, a spin structure, and therefore according to Lawson and Michelsohn (1989) has zero first Chern class. Thus the

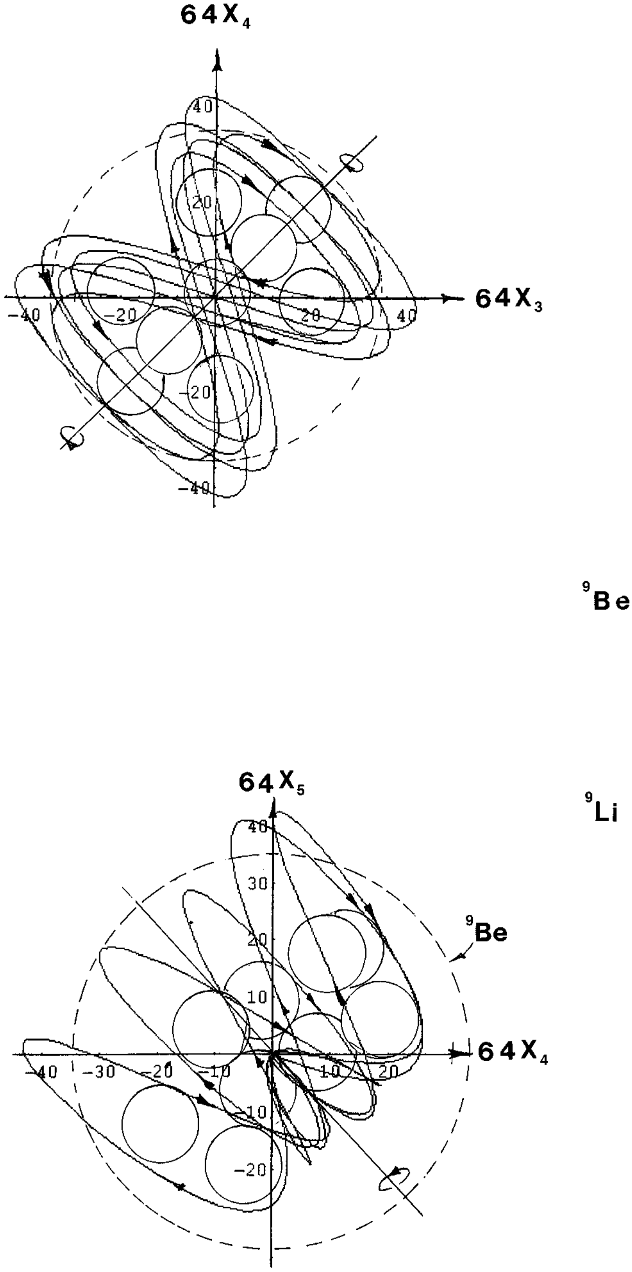

Fig. 1. Geodesics or strings on ^{9}Be, ^{9}Li.

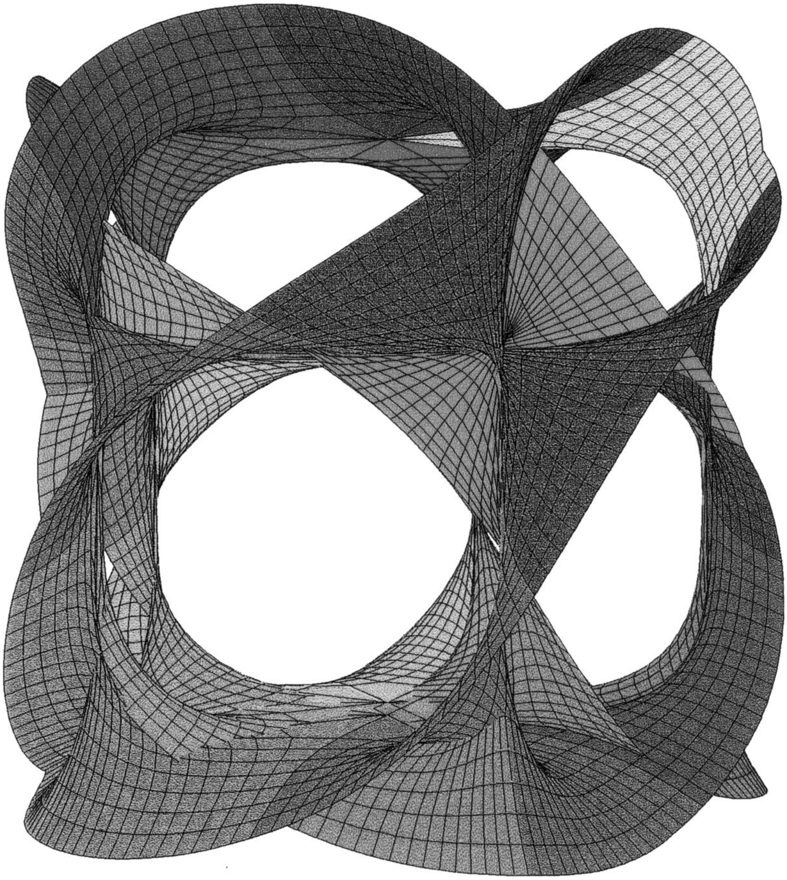

Fig. 2. Hanson quintic hypersurface.

internal Kaehler space is also a Calabi-Yau space K and it is possible to find a decomposition $M^4 \times K$ of 10-dimensional space-time for each odd-A nucleus. One may say that the spinorial representations of $SO(10)$ describe the particles governed by representations of $SO(4)$ that lie in K. A resume' of the nuclear representations of $SO(4)$ follows in the next section.

2. NUCLEAR REPRESENTATIONS OF *O*(4)

We begin with an irreducible self-representation

$$\frac{1}{4}\Psi = (iE_4\Psi_1 + E_{23}\Psi_2 + E_{14}\Psi_3 + E_{05}\Psi_4)e \tag{2.1}$$

of the centralizer D of the Dirac ring where Eddington's E-numbers are related to the Dirac matrices by

$$\gamma_\nu = iE_{o\nu}, \quad E_{\mu\nu} = E_{\rho\mu}E_{\rho\nu} = -E_{\nu\mu}, \quad E_{\mu\nu}^2 = -1, \quad \mu < \nu = 1, \ldots, 5$$

and the commuting operators E_{23}, E_{14}, and E_{05} respectively are independent rotations in 3-space, 4-space, and isospace that correspond to the spin σ, parity π, and charge T_3 carried by a single-nucleon. The parameters Ψ_2, Ψ_3, Ψ_4 are half-angles of rotation and e is a primitive idempotent; E_4 is the unit matrix. To see how E_{14} is related to parity we notice that a rotation through π about t will send x to $-x$ without inverting time but instead changing to a left-handed coordinate system. The operators of the centralizer obey the multiplication table:

	E_{23}	E_{14}	E_{05}
E_{23}	i^2	iE_{05}	iE_{14}
E_{14}	iE_{05}	i^2	iE_{23}
E_{05}	iE_{14}	iE_{23}	i^2

(2.2)

A many-nucleon representation is found in the enveloping algebra $A(\gamma)$ of the Dirac ring by constructing tensor products of (2.1) with itself. The basis elements are the $4^A \times 4^A$ matrices

$$E_{\mu\nu}^l = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4 \tag{2.2a}$$

with $E_{\mu\nu}^l$ in the 1st position. The elements $E_{\mu\nu}^l$, $E_{\mu\nu}^{(l+1)}$ commute, and $A(\gamma)$ is found to have the following generators, or de Broglie operators

$$\Gamma_\nu^{(A)} = \frac{1}{2}\left(E_{0\nu}^1 + E_{0\nu}^2 + \cdots + E_{o\nu}^A\right); \quad \nu = 1, \ldots, 5 \tag{2.3a}$$

$$\sigma_{\mu\nu}^{(A)} = \left[\Gamma_\mu^{(A)}, \Gamma_\nu^{(A)}\right] = \frac{1}{2}\left(E_{\mu\nu}^1 + \cdots + E_{\mu\nu}^A\right); \quad \mu < \nu = 1, \ldots, 5 \tag{2.3b}$$

$$\begin{aligned} \eta_\nu^{(A)} &= E_{0\nu} \otimes \cdots \otimes E_{0\nu} = E_{0\nu}^1 E_{0\nu}^2 \cdots E_{0\nu}^A, \\ \eta_{\mu\nu}^{(A)} &= \eta_\mu^{(A)} \eta_\nu^{(A)} = E_{\mu\nu}^1 E_{\mu\nu}^2 \cdots E_{\mu\nu}^A \end{aligned} \tag{2.3c}$$

Then by using the 4×4 matrix representations of E_{23}, E_{14}, and E_{05} one can find fibers consisting of all those states that have the same quantum numbers of spin, parity, and charge. The fiber bundle space has a beautiful de Rham decomposition into isobaric multiplets each of which is characterized by the matrix representations

$$\sigma_i = E_N \otimes {}^P\Gamma_i + {}^N\Gamma_i \otimes E_P, \quad \pi_i = E_N \otimes {}^P\Gamma_i - {}^N\Gamma_i \otimes E_P, \quad i = 1,\ 2,\ 3 \tag{2.4}$$

where ${}^P\Gamma_i$, ${}^N\Gamma_i$ are $(P+1)$-, $(N+1)$-dimensional Lie operators of $SO(3)$ and E_P, E_N are $(P+1)$-, $(N+1)$-dimensional unit matrices (cf. de Wet, 1973). The operator σ_i may be recognized as the well-known angular momentum matrix for a coupled system of P protons and N neutrons and π_i is a parity operator.

In fact σ_i, π_i are the generators of $SO(4)$ and the fibration introduces a Yang-Mills field with connections on a fiber bundle (cf. for example, Schwarz, 1991, Introduction).

The irreducible representations or minimal left ideals of $A(\gamma)$ are

$$\Psi^{(A)} = \sum_{\lambda} C_{[\lambda]} P_{[\lambda]} \tag{2.5}$$

where $P_{[\lambda]}$ is a projection operator

$$\begin{aligned} P_{[\lambda]} = {} & i^{-A}\big(i^A \psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} + \eta_{23}^{(A)}\psi_1^{\lambda_2}\psi_2^{\lambda_1}\psi_3^{\lambda_4}\psi_4^{\lambda_3}\big)\epsilon_A \\ & + i^{-A}\big(\eta_{14}^{(A)}\psi_1^{\lambda_3}\psi_2^{\lambda_4}\psi_3^{\lambda_1}\psi_4^{\lambda_2} + \eta_5^{(A)}\psi_1^{\lambda_4}\psi_2^{\lambda_3}\psi_3^{\lambda_2}\psi_4^{\lambda_1}\big)\epsilon_A \end{aligned} \tag{2.6a}$$

satisfying

$$P_{[\lambda]}^2 = P_{[\lambda]}\psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} \tag{2.6b}$$

where $\epsilon_A = e \otimes \cdots \otimes e$ is idempotent so that (2.6b) has the form (2.1) in configuration space. Examination of (2.6a) shows that, in view of the canonical labeling scheme, the first two terms characterize a given nucleus while the other pair characterize its mirror ($T \to -T$). Furthermore, if $[\lambda_1\lambda_2\lambda_3\lambda_4]$ is characterized by σ_0, π_0 its partner $[\lambda_2\lambda_1\lambda_4\lambda_3]$ will be characterized by $-\sigma_0, -\pi_0$ so as will be confirmed $C_{[\lambda_1\lambda_2\lambda_3\lambda_4]} = -C_{[\lambda_2\lambda_1\lambda_4\lambda_3]}$ are both acceptable. In this way there is a sign ambiguity that can be settled only by comparison with the eigenvalues of an irreducible subspace μ of $C_{[\lambda]}$.

Now by virtue of the isomorphism between Clifford multiplication and the exterior product (Lawson and Michelsohn, 1989)

$$C_{[\lambda]} = i^{\lambda_1} \sum \big(E_{23}^1 \cdots E_{23}^{\lambda_2} E_{14}^{\lambda_2+1} \cdots E_{14}^{\lambda_2+\lambda_3} E_{05}^{\lambda_2+\lambda_3+1} \cdots E_{05}^{A-1}\big) \tag{2.7}$$

is a threeform in the centralizer D. Here there is summation of all the $N_{[\lambda]} = A!/(\lambda_1!\lambda_2!\lambda_3!\lambda_4!)$ combinations of the basis elements and by (2.5) a matrix representation of the many nucleon problem will have the rows of $C_{[\lambda]}$ labelled by the states $[\lambda]$. However, a Hodge decomposition of the central equation (2.7) may be obtained without any appeal to matrices and also the eigenvalues that determine the metric may be found up to sign from (1.1). The fact that signs can be chosen to agree exactly with a matrix representation based on (2.4) for the light nuclei up to ^{11}C justifies the canonical labeling adopted. In the case of ^{13}C there is a tiny spin mutation of 1/450 in two paired states which suggests a reformulation of the Lie commutation relations for these states as employed in quantum group theory. Santilli (1992) called these algebras Lie-Admissible. They indicate $SO(10)$ symmetry breaking.

To find the Hodge decomposition of (2.7) we write it

$$C_{[\Lambda]} = i^{\Lambda_1}\sigma_o^{\Lambda_2}\pi_o^{\Lambda_3}T_o^{\Lambda_4} - \sum_{\lambda} i^{\lambda_1}\sigma_o^{\lambda_2}\pi_o^{\lambda_3}T_o^{\lambda_4} \tag{2.7a}$$

where $[\Lambda] = [\Lambda_1\Lambda_2\Lambda_3\Lambda_4]$ is the ground state and

$$\sigma_o = 2\sigma_1 = \left(E_{23}^1 + \cdots + E_{23}^A\right) = 2is, \quad \pi_o = 2\pi_1 = \left(E_{14}^1 + \cdots + E_{14}^A\right) = 2ip$$

$$T_o = 2\Gamma_5^{(A)} = \left(E_{05}^1 + \cdots + E_{05}^A\right) = i(Z - N) = 2T_3 \tag{2.8}$$

The summation contains all those terms arising from repeated indices, e.g., $E_{23}^k E_{23}^k$, $E_{23}^k E_{14}^k$, $E_{23}^k E_{05}^k$, and $E_{14}^k E_{05}^k$ that yield a single term according to (2.2) and (2.2a).

An elementary application of (2.7a) is

$$\sigma_o T_o = P\left(E_{23}^j E_{05}^i\right) + i\pi_o \tag{2.9a}$$

where P signifies summation over the $A!/(A-n)!$ permutations of the n generators in the bracket. Then

$$C_{[(A-2)101]} = i^{(A-2)} P\left(E_{23}^i E_{23}^j\right) = i^{(A-2)}(\sigma_o T_o - i\pi_o) \tag{2.9b}$$

and if $A = 3$, $Z = 1$, $T_o = i(Z - N) = -i$

$$C_{[1101]} = (\sigma_o + \pi_o) \tag{2.9c}$$

which characterizes the ground state of ^{3}H. The ground state of ^{3}He is obtained by interchanging $\sigma_o \Leftrightarrow \pi_o$ in (2.9b) to get

$$C_{[(A-2)011]} = i^{(A-2)}(\sigma_o T_o - i\sigma_o)$$

so that if $A = 3$, $Z = 2$, $T_o = i$, we find the mirror nucleus

$$C_{[1101]} = (\sigma_o - \pi_o) \tag{2.9d}$$

which is manifestly *CP*-invariant because $T_o \to -T_o$ is accompanied by $\pi \to -\pi_o$.

From (2.4) we find the dual complex spaces

$$\sigma_o = \begin{bmatrix} & X \\ -X^{\mathrm{T}} & \end{bmatrix}, \quad \pi_o = \begin{bmatrix} & X^{\mathrm{T}} \\ -X & \end{bmatrix}, \quad X = \begin{bmatrix} -1 & -\sqrt{2} & o \\ \sqrt{2} & 1 & -\sqrt{2} \\ o & \sqrt{2} & -1 \end{bmatrix} \tag{2.10}$$

which lie in a Kaehler manifold known to have a Hodge decomposition (cf. for example, Griffiths and Harris, 1978). Thus following Kobayashi and Nomizu (1969, Ch. 9) we set π_0 equal to the subspace $\Lambda^{1,0}$ of (1,0) forms, σ_o equal to the subspace $\Lambda^{0,1}$, then $C_{[1101]}$ is the Hodge decomposition $H^{1,0} + H^{0,1}$.

The eigenvalues of $(\sigma_o + \pi_o)$ found from the labeling (1.1) are $[-2i; 2i; -2i; 2i; -2i; 2i]$ corresponding to the states

([0210]; [2001]; [2010]; [0201]; [1110]; [1101])

These are in one-to-one correspondence with the matrix representation (2.10) but the latter specifies the sets

$$(2i; -2i; 2i), \quad -(2i; -2i; 2i)$$

In general we will find a decomposition into (p, q) forms whenever the threeform $\sigma_o \pi_o T_o$ contains terms with the same indices (as shown by (2.9a) where π_o arises from all the products $E^i_{23} E^i_{14}$). There will be coupling constants that count the number of times an irreducible representation occurs. The process beginning with (2.9) may be continued by "adding" one nucleon at a time, i.e., by multiplying by σ_o, π_o, T_o until $\Lambda_2 + \Lambda_3 + \Lambda_4 = A - \Lambda_1$ and in this way we find the $A = 9$ operators

$$^9\text{Li}: C_{[3303]} = \frac{i^3 P\left(E^i_{23} E^j_{23} E^k_{23} E^l_{05} E^m_{05} E^n_{05}\right)}{(3!3!)}$$

$$= \frac{1}{6}\left[34(\sigma_0 + \pi_0) + 9\left(\sigma_0 \pi_0^2 + \sigma_0^2 \pi_0\right) + \left(\sigma_0^3 + \pi_0^3\right)\right] \quad (2.11a)$$

$$^9\text{C}: C_{[3033]} = \frac{1}{6}\left[34(\sigma_0 - \pi_0) + 9\left(\sigma \pi_0^2 - \sigma_0^2 \pi_0\right) + \left(\sigma_0 - \pi_0^3\right)\right] \quad (2.11b)$$

$$^9\text{B}: C_{[3123]} = -\frac{1}{2}\left[34(\sigma_0 + \pi_0) + \left(\sigma_0 \pi_0^2 + \sigma_0^2 \pi_0\right) + \left(\sigma_0^3 + \pi_0^3\right)\right] \quad (2.12a)$$

$$^9\text{Be}: C_{[3213]} = -\frac{1}{2}\left[34(\sigma_0 - \pi_0) + \left(\sigma_0 \pi_0^2 - \sigma_0^2 \pi_0\right) + \left(\sigma_0^3 - \pi_0^3\right)\right] \quad (2.12b)$$

which are *CP*-invariant and may be expressed in the matrix form (2.10) by means of (2.4). The representations of mirror nuclei are identical up to an equivalence transformation of rows and columns.

Equations (2.11) and (2.12) are in terms of the harmonics

$$H^{1,0} + H^{0,1} = (\sigma_0 + \pi_0); \quad H^{2,1} + H^{1,2} = \left(\sigma_0^2 \pi_0 + \sigma_0 \pi_0^2\right);$$
$$H^{3,0} + H^{0,3} = \left(\sigma_0^3 + \pi_0^3\right) \quad (2.13)$$

which is a Hodge decomposition of the cohomology of the Kaehler manifold consisting of classes that are closed but not exact as confirmed by Kobayashi and Nomizu who show that the exterior derivative of an odd form is even. Thus the observation that (2.11) and (2.12) contain no even forms proves that all classes are closed. Apart from the contribution of Griffiths (1969); Green *et al.* (1988, Section 16.3.2) show that the harmonics (2.13) may characterize a quintic hypersurface illustrated by Fig. 2, although this need not always be the case as will become clear when a metric is introduced.

We will need to exponentiate an irreducible subspace μ of the matrix representation of $C_{[\Lambda]}$. It has the structure (2.10) where X is now a real symmetric $p \times p$ matrix A with coordinates $k = \gamma_k t$ and there is a one-to-one correspondence between the eigenvalues γ_k and the state $[\lambda]_k = [\lambda_1\lambda_2\lambda_3\lambda_4]_k$. The exponential formula (de Wet, 1996) states that

$$e^{\mu t} = \mu \sum_{k=0,1}^{n} \frac{F_k(\mu)\cos\gamma_k t}{i\gamma_k F(i\gamma_k)} + i \sum_{k=1,2}^{n} \frac{F_k(\mu)\sin\gamma_k t}{F_k(i\gamma_k)} \tag{2.14}$$

where

$$F(\mu) = \mu(\mu^2+1)(\mu^2+\gamma_2^2)\cdots(\mu+\gamma_n^2) = 0$$

$$F_0(\mu) = F(\mu)/\mu, \quad F_k(\mu) = F(\mu)/(\mu^2+\gamma_k^2), \quad F_j(\mu)F_k(\mu) = 0 \tag{2.14a}$$

and

$$K_k(\mu) = \frac{i\gamma_k F_k(\mu)}{\mu F_k(i\gamma_k)} \tag{2.15}$$

is idempotent. Thus (2.14) follows by differentiating at $t = 0$ because $\sum_k K_k(\mu)$ is a decomposition of unity. Also $e^{\mu t}$ is orthogonal and unimodular because

$$e^{\mu t}(e^{\mu t})^{\mathrm{T}} = e^{(\mu+\mu^{\mathrm{T}})t} = 1 = e^{\mathrm{Tr}\,\mu t} = Det\, e^{\mu t} \tag{2.16}$$

In this way (2.14) is an irreducible representation of $SO(4)$ that generates an internal nuclear space that is Calabi-Yau. A metric, intimately associated with the wave function, is found by writing (2.14)

$$e^{\mu t} = Z_0(\cos t) + Z_1(\sin t) = \boxed{\begin{matrix} Z_0 & Z_1 \\ Z_1 & Z_0 \end{matrix}}$$

and using the formula of Wong (1967)

$$ds^2 = \mathrm{Tr}\frac{dT}{(1+TT^{\mathrm{T}})}\frac{d\bar{T}^{\mathrm{T}}}{(1+TT^{\mathrm{T}})} \tag{2.17}$$

where

$$T = Z_1 Z_0^{-1} = -T^{\mathrm{T}} = \mu \sum_{k=1,2}^{n} \frac{i(F_\mu(\mu)/\mu)\tan\gamma_k t}{F_\mu(i\lambda_k)} \tag{2.17a}$$

$$T\bar{T}^t = \sum_{k=1,2}^{n} K_k(\mu)\tan^2\gamma_k t \tag{2.17b}$$

Here $\bar{T}^{\mathrm{T}}$ and $d\bar{T}^{\mathrm{T}}$ are conjugate transposes of T and dT and (2.17) reduces to the flat measure carried by a torus, namely

$$ds^2 = \sum_{k=1,2}^{n} dz_k\, d\bar{z}_k, \qquad z_k = i\gamma_k t \tag{2.18}$$

which is the Kaehler condition that the metric approximate the Euclidean metric to Order 2 at each point (Griffiths and Harris, 1978, Ch. 0.7).

However, (2.14a) depends on a translation to a normal canonical form

$$(1; \gamma_2; \ldots; \gamma_n), \qquad n \leqslant p \tag{2.19}$$

where $(\gamma_2, \ldots, \gamma_n)$ are all positive. If this condition is not met we must add an angular momentum γ_0 equal to the greatest negative γ and then divide by $\gamma_f = (\gamma_k + \gamma_0)$, which may be absorbed in t and does not change the geodesics although there is a frequency change in the wave functions $e^{\mu t}$. The effect of the translation is to introduce a "twist" $e^{\mu t}\, e^{i\gamma_0 t}$ that multiplies (2.17a) by $\tan^2 \gamma_0 t$ and leads to a distorted metric

$$\begin{aligned} ds^2 &= \sum_k g(\gamma_k) g(-\gamma_k t)\, d(\gamma_t t)\, d(-\gamma_k t) \\ &= \sum_k \frac{\tan^2 \gamma_0 t \sec^4 \gamma_k t}{(1 + \tan^2 \gamma_0 t \tan^2 \gamma_k t)^2}\, dz_k\, d\bar{z}_k = \sum_k g_{k\bar{k}}\, dz_k\, d\bar{z}_k \end{aligned} \tag{2.20}$$

which is independent of μ because of the idempotent factor (2.15) and reduces to (2.18) when $\tan \gamma_0 t = 1$. Here $k = \gamma_k t$, $\bar{k} = -\gamma_k t$ are respectively coordinates of $X \to A$, $-X^{\mathrm{T}} \to -A$ in (2.10), while $i\gamma_k t$ are the coordinates of μ.

Finally because by (2.14a), a k-plane is annihilated by γ_k, its orientation is determined by the remaining planes. In this way a spinor field corresponding to the state $[\lambda]_k$ and propagated only around the section $k\bar{k}$ will return to its original value that is precisely the condition given by Green *et al.* (1988, Section 15.1.3) for a Calabi-Yau space to have $SU(3)$ holonomy.

3. NUCLEAR SHAPES

In this section we will use (2.14) to determine the nuclear shapes of Fig. 1 by finding the geodesics (or strings according to Green *et al.*, 1988, Ch. 1) on the manifolds of Li-9, Be-9, and their mirror partners. In the case of Be-9

$$\mu = \begin{array}{c|c|c|} & k & \bar{k} \\ \hline k & & A \\ \hline \bar{k} & -A & \\ \hline \end{array} \tag{3.1}$$

where the real bisymmetric matrix A, with normalized eigenvalues

$$(0; 1/2; 1; 5/2; 5) \tag{3.2}$$

has been obtained by interchanging rows and columns. There is no twist so the metric (2.18) is flat. Preferred central states are chosen with elements

$$A_{43} = -\frac{3}{4}, \qquad A_{44} = \frac{9}{4}. \tag{3.3}$$

After evaluating $[A^m]_{43}$, $[A^m]_{44}$ for $m = 3, 5, 7$, Equation (2.14) yielded the wave functions

$$X_3 = 1/16(-\sin t/2 + \sin t + 5\sin 5t/2 - 5\sin 5t) \tag{3.4a}$$

$$X_4 = X_3(2\pi - t) = 1/16(-\sin t/2 - \sin t + 5\sin 5t/2 + 5\sin 5t) \tag{3.4b}$$

and it may readily be confirmed that at $t = 0$, $dX/dt = -3/4$, $dX/dt = 9/4$ in agreement with (3.3). Because $X_4 = X_3(2\pi - t)$ the wave functions are complementary and consequently generate closed geodesics in 3-space (cf. Kobayashi and Nomizu, 1969, Ch. 9). They may be thought of as electric flux lines enclosing the rotating and spinning nucleons as depicted in Fig. 1 (cf. t'Hooft, 1979). However, because the nucleus is pulsating and rotating, nucleons sketched in the figure will move as the geodesic proceeds round them so only average positions can be shown.

Although the complementary wave functions are not given, Fig. 3 shows the geodesics on ^{13}C (to the same scale). There are two nucleons outside of a central core and the axis of rotation is perpendicular to the figure. There is also a twisted metric but no singularities at the origin. ^{9}Be and ^{13}C are dipoles.

Turning now to ^{9}Li, the central matrix elements of preferred states are

$$A_{44} = \frac{31}{12}, \qquad A_{45} = -1 \tag{3.5}$$

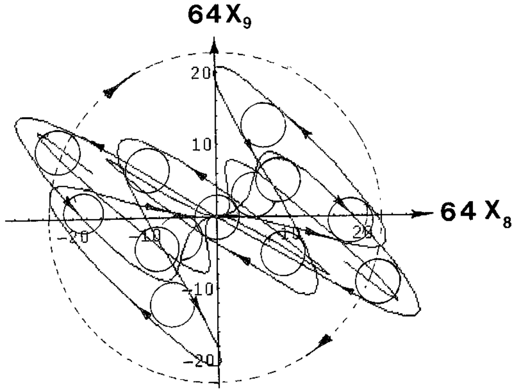

Fig. 3. Geodesics on C-13.

and the normalized eigenvalues of A are

$$\left(0; \frac{1}{2}; \frac{5}{6}; 1; \frac{4}{3}; \frac{3}{2}; \frac{5}{2}; 5\right) \tag{3.6}$$

with a twist of $\gamma_0 = 5/3$. After evaluating $[A^m]_{44}$, $[A^m]_{45}$, for $m = 3, 5, 7, \ldots, 13$, Equation (2.14) yielded the complementary wave functions

$$X_4 = \frac{1}{64}\sin\frac{5t}{3}\left(9\sin\frac{t}{2} + 5\sin\frac{5t}{6} + \frac{3}{2}\sin t + \frac{1}{2}\sin\frac{4}{3}t + 3\sin\frac{3}{2}t \right.$$
$$\left. + 15\sin\frac{5}{2}t + 22.5\sin 5t\right) \tag{3.7a}$$

$$X_5 = +X_4(6\pi - t) = \frac{1}{64}\sin\frac{5}{3}t\left(9\sin\frac{t}{2} + 5\sin\frac{5t}{6} - \frac{3}{2}\sin t \right.$$
$$\left. -\frac{1}{2}\sin\frac{4}{3}t + 3\sin\frac{3}{2}t + 15\sin\frac{5}{2}t - 22.5\sin 5t\right) \tag{3.7b}$$

which are also plotted to approximately the same scale in Fig. 1. Again at $t = 0$, dX_4/dt, dX_5/dt satisfy (3.5).

The Lithium nucleus rotates into itself after $t = 6\pi$, and Fig. 4 shows the twisted measure (derived from (2.20)) namely

$$g_{\kappa\bar{\kappa}} = \frac{\tan^2(5/3)t\,\sec^4(5/6)t}{(1 + \tan^2(5/3)t\,\tan^2(5/6)t)^2} \tag{3.8}$$

on the preferred plane $\gamma_k = \gamma_0/2 = 5/6$ over this range. The troughs correspond to the small metric of Fig. 2, as derived in the Appendix, and apart from oscillations to the flat metric $g_{k\bar{k}} = 1$, corresponding to $\tan\frac{5}{3}t = 1$, there are five singularities at the origin which occur when $\sin\gamma_0 t = \tan\gamma_0 t = 0$, i.e., when $t = \frac{3\pi}{5}, \frac{9\pi}{5}, 3\pi, \frac{21\pi}{5}, \frac{27\pi}{5} < 6\pi$. These define a pole of order 5, that may be reduced to a minimal

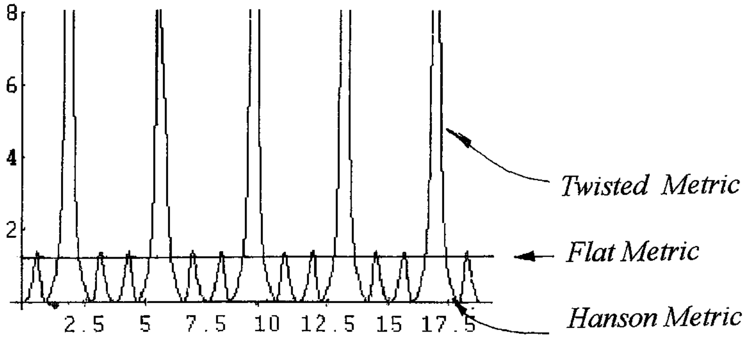

Fig. 4. Nuclear metrics.

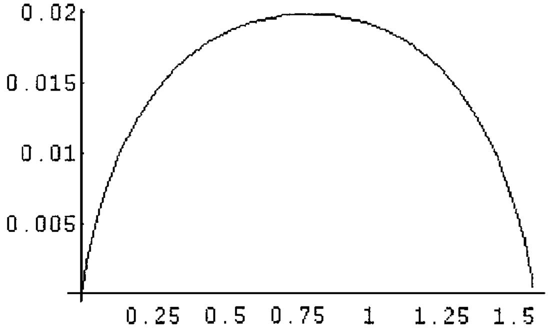

Fig. 5. Hanson metric.

order $n(q) = 4$, which Griffiths (1969, Table 1) demonstrates can generate at least a quintic algebraic variety V in the complex projective space P^4. Moreover, the Hodge decomposition must be associated with a three-form H^3.

Because of the absence of singularities in the Hanson metric (shown in Fig. 5) Fig. 2 can only be regarded as a first approximation to the manifold of ^{9}Li. In fact there are 101 possible quintic hypersurfaces and one must seek that one with the metric (3.8). The hypersurface considered in the Appendix is described by two parameters ξ, t. Thus to find that part defining a geodesic, or one-parameter subgroup, we set $\xi = 0$, $n = 5$ to get Fig. 5. The metric is tiny.

APPENDIX

Hanson (1994) looked at a three-dimensional section

$$z_1^n + z_2^n = 1 \tag{A1}$$

of the abelian variety

$$z_1^n + z_2^n + z_3^n + z_4^n = 1$$

in complex projective 4-space P. A two-dimensional solution of (A1) is

$$z_1(t, \xi, k_1) = s(k_1, n)u_1(t, \xi)^{2/n}$$
$$z_2(t, \xi, k_2) = s(k_2, n)u_2(t, \xi)^{2/n}$$

where

$$u_1(t, \xi) = \frac{1}{2}(\exp(\xi + it) + \exp(-\xi - it))$$
$$u_2(t, \xi) = \frac{1}{2i}(\exp(\xi + it) - \exp(-\xi - it)), \tag{A2}$$

and

$$s(k, n) = \exp(2\pi ik/n)$$

is a phase factor consisting of the nth root of unity for the integers $0 \leq k \leq (n-1)$. Then

$$dz^1 = s(k_1, n)\left(\frac{2}{n}u_1^{\frac{2}{n}-1}\right)\left(\frac{\partial u_1}{\partial t}\,dt + \frac{\partial u_1}{\partial \xi}d\xi\right) = s(k_1, n)\frac{2}{n}u_1^{\frac{2}{n}-1}u_2(-dt + id\xi)$$

$$dz_1\,d\overline{z}_1 = \frac{4}{n^2}(u_1\overline{u}_1)^{\frac{2}{n}-1}u_2\overline{u}_2(dt^2 + d\xi^2)$$

$$dz_2 = s(k_2, n)\frac{2}{n}u_2^{\frac{2}{n}-1}u_1(dt - id\xi)$$

$$dz_2\,d\overline{z}_2 = \frac{4}{n^2}(u_2\overline{u}_2)^{\frac{2}{n}-1}u_1\overline{u}_1(dt^2 + d\xi^2)$$

and the Fubini-study metric given by Kobayashi and Nomizu (1969, Ch. 9) reads

$$\begin{aligned} ds^2 &= \frac{dz_1\,d\overline{z}_1 + dz_2\,d\overline{z}_2 - (z_1\,dz_2 - z_2\,dz_1)(\overline{z}_1\,d\overline{z}_2 - \overline{z}_2\,d\overline{z}_1)}{(1 + z_1\overline{z}_1 + z_2\overline{z}_2)^2} \\ &= \frac{4((u_1\overline{u}_1)^{2-\frac{2}{n}} + (u_2\overline{u}_2)^{2-\frac{2}{n}} - 1)(dt^2 + d\xi^2)}{n^2[(u_1\overline{u}_1)(u_2\overline{u}_2)]^{1-\frac{2}{n}}\left(1 + (u_1\overline{u}_1)^{\frac{2}{n}} + (u_2\overline{u}_2)^{\frac{2}{n}}\right)^2} \\ &= G(dt^2 + d\xi^2) \end{aligned} \tag{A3}$$

where from (A2)

$$(u_1\overline{u}_1) = \frac{1}{2}(\cosh 2\xi + \cos 2t), \quad (u_2\overline{u}_2) = \frac{1}{2}(\cosh 2\xi - \cos 2t), \quad u_1^2 + u_2^2 = 1$$

Equation (A3) is plotted in Fig. 5 with $\xi = 0$, and $n = 5$.

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